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On Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals

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ABSTRACT. In this paper, we have established some generalized integral inequalities of Hermite-Hadamard-Fejér type for generalized fractional integrals. The results presented here would provide generalizations of those given in earlier works.

1. INTRODUCTION

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function define on an interval I of real numbers, and $a, b \in I$ with $a < b$. Then the following inequalities hold:

$$f\left(\tfrac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

It was first discovered by Hermite in 1881 in the Journal Mathesis. This inequality (1) was nowhere mentioned in the mathematical literature untill 1893. In [4], Beckenbach, a leading expert on the theory of convex functions, wrote that the inequality (1) was proved by Hadamard in 1893. In 1974, Mitrinovič found Hermite and Hadamard's note in Mathesis. That is why, the inequality (1) was known as Hermite-Hadamard inequality. We note that Hermite-Hadamard's inequality may be regarded as a refinements of the concept of convexity and it follows easily from Jensen's inequality. This inequality (1) has been received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [4]-[16], [18], [20].

The most well known inequalities connected with the integral mean of a convex functions are Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejěr inequality.

In [10], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequalities (1):

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Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a convex on I and let $a, b \in I$ with $a < b$. Then the inequality*

$$f\left(\tfrac{a+b}{2}\right) \int_a^b g(x)\mathrm{d}x \leq \frac{1}{b-a} \int_a^b f(t)g(x)\mathrm{d}t \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)\mathrm{d}x. \tag{2}$$

holds, where $f : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric to $\dfrac{a+b}{2}$.

In [11], Sarikaya et al. represented Hermite–Hadamard's inequalities in fractional integral forms as follows.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals holds*

$$f\left(\tfrac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{3}$$

with $\alpha > 0$.

In [5] Set et. al. obtained the following lemma.

Lemma 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and let $g : [a, b] \to \mathbb{R}$. If f', $g \in L[a, b]$, then the following identity for fractional integrals holds:*

$$\begin{aligned} &f\left(\tfrac{a+b}{2}\right)\left[J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} g(b)\right] \\ &- \left[J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} (gf)(a) + J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} (fg)(b)\right] = \frac{1}{\Gamma(\alpha)} \int_a^b k(t)\mathrm{d}f(t), \end{aligned} \tag{4}$$

where

$$k(t) = \begin{cases} \displaystyle\int_a^t (s-a)^{\alpha-1} g(s)\mathrm{d}s, & t \in \left[a, \dfrac{a+b}{2}\right], \\ \displaystyle\int_b^t (b-s)^{\alpha-1} g(s)\mathrm{d}s, & t \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.1. Let $h(x)$ be an increasing and positive monotone function on $[0, \infty)$, also derivative $h^{'}(x)$ is continuous on $[0, \infty)$ and $h(0) = 0$.

The space $X_h^p(0,\infty)$ $(1 \le p < \infty)$ of those real-valued Lebesque measurable functions f on $[0,\infty)$ for which

$$(5) \qquad ||f||_{X_h^p} = \left(\int_0^\infty |f(t)|^p \, h'(x)\, dt \right)^{\frac{1}{p}} < \infty, \quad 1 \le p \le \infty$$

and for the case $p = \infty$

$$(6) \qquad ||f||_{X_h^\infty} = \text{ess} \sup_{1 \le t < \infty} \left[f(t) h'(x) \right].$$

Definition 1.2. ([6]). In particular, when $h(x) = x$ $(1 \le p < \infty)$ the space $X_h^p(0,\infty)$ coincides with the $L_p[0,\infty)-$space ($||f||_{X_h^\infty} = ||f||_\infty$) and also if we take $h(x) = \dfrac{x^{k+1}}{k+1}$ $(k \ge 0)$ the space $X_h^p(0,\infty)$ coincides with the $L_{p,k}[0,\infty)-$space.

Definition 1.3. ([1]). Let (a,b) be a finite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $h(x)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $h'(x)$ on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function h on $[a,b]$ are defined by

$$(7) \qquad \left(J_{a^+,h}^\alpha f \right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[h(x) - h(t) \right]^{\alpha-1} h'(t) f(t)\, dt, \quad x \ge a$$

and

$$(8) \qquad \left(J_{b^-,h}^\alpha f \right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[h(t) - h(x) \right]^{\alpha-1} h'(t) f(t)\, dt, \quad x \le b.$$

Definition 1.4. If we take $h(x) = x$, then the equalities (7) and (8) will be

$$(9) \qquad \left(J_{a^+}^\alpha f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a$$

and

$$(10) \qquad \left(J_{b^-}^\alpha f \right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x.$$

These integrals are called left-sided Riemann-Liouville fractional integral and right-sided Riemann-Liouville fractional integral respectively [1]-[3], [6], [17].

In this paper, we have established some generalized fractional integral inequalities. The results presented here would provide generalizations of those given in earlier works.

2. Main Results

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and let $g : [a,b] \to \mathbb{R}$. If f', $g \in X_h^p[a,b]$, then the following identity for fractional integrals holds:*

$$
\begin{aligned}
& f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(h(a)) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(h(b))\right] \\
& \quad - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\left(g \times (f \circ h)\right)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\left(g \times (f \circ h)\right)(b)\right] \qquad (11) \\
& = \frac{1}{\Gamma(\alpha)} \int_a^b k(t) \mathrm{d}f(h(t))
\end{aligned}
$$

where

$$
k(t) = \begin{cases} \displaystyle\int_a^t (h(s)-h(a))^{\alpha-1} g(s) h'(s) \mathrm{d}s, & t \in \left[a, \dfrac{a+b}{2}\right], \\ \displaystyle\int_b^t (h(b)-h(s))^{\alpha-1} g(s) h'(s) \mathrm{d}s, & t \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}
$$

Proof. It suffices to note that

$$
\begin{aligned}
I &= \int_a^b k(t) \mathrm{d}f(h(t)) \\
&= \int_a^{\frac{a+b}{2}} \left(\int_a^t (h(s)-h(a))^{\alpha-1} g(s) h'(s) \mathrm{d}s \right) \mathrm{d}f(h(t)) \\
&\quad + \int_{\frac{a+b}{2}}^b \left(\int_b^t (h(b)-h(s))^{\alpha-1} g(s) h'(s) \mathrm{d}s \right) \mathrm{d}f(h(t)) \\
&= I_1 + I_2.
\end{aligned}
$$

By integration by parts, we get

$$
\begin{aligned}
I_1 &= \int_a^{\frac{a+b}{2}} \left(\int_a^t (h(s)-h(a))^{\alpha-1} g(s) h'(s) \mathrm{d}s \right) \mathrm{d}f(h(t)) \\
&= \left(\int_a^t (h(s)-h(a))^{\alpha-1} g(s) h'(s) \mathrm{d}s \right) f(h(t)) \Bigg|_a^{\frac{a+b}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&- \int_a^{\frac{a+b}{2}} (h(t)-h(a))^{\alpha-1} g(t) f(h(t)) h'(t) \mathrm{d}t \\
&= f\left(h\left(\tfrac{a+b}{2}\right)\right) \int_a^{\frac{a+b}{2}} (h(s)-h(a))^{\alpha-1} g(s) h'(s) \mathrm{d}s \\
&\quad - \int_a^{\frac{a+b}{2}} (h(t)-h(a))^{\alpha-1} g(t) f(h(t)) h'(t) \mathrm{d}t \\
&= \Gamma(\alpha) \left[f\left(h\left(\tfrac{a+b}{2}\right)\right) J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} g(h(a)) - J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} \left(g\left(f\circ h\right)\right)(a) \right]
\end{aligned}
$$

and similarly

$$
\begin{aligned}
I_2 &= \int_{\frac{a+b}{2}}^{b} \left(\int_b^t (h(b)-h(s))^{\alpha-1} g(s) h'(s) \mathrm{d}s \right) \mathrm{d}f(h(t)) \\
&= \left(\int_b^t (h(s)-h(a))^{\alpha-1} g(s) h'(s) \mathrm{d}s \right) f(h(t)) \Bigg|_{\frac{a+b}{2}}^{b} \\
&\quad - \int_{\frac{a+b}{2}}^{b} (h(b)-h(t))^{\alpha-1} g(t) f(h(t)) h'(t) \mathrm{d}t \\
&= f\left(h\left(\tfrac{a+b}{2}\right)\right) \int_{\frac{a+b}{2}}^{b} (h(b)-h(s))^{\alpha-1} g(s) h'(s) \mathrm{d}s \\
&\quad - \int_{\frac{a+b}{2}}^{b} (h(b)-h(t))^{\alpha-1} g(t) f(h(t)) h'(t) \mathrm{d}t \\
&= \Gamma(\alpha) \left[f\left(h\left(\tfrac{a+b}{2}\right)\right) J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} g(h(b)) - J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} \left(g\left(f\circ h\right)\right)(b) \right].
\end{aligned}
$$

Thus, can write

$$
\begin{aligned}
I &= I_1 + I_2 \\
&= \Gamma(\alpha) \Bigg\{ f\left(h\left(\tfrac{a+b}{2}\right)\right) \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} g(h(a)) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} g(h(b)) \right] \\
&\quad - \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} \left(g\left(f\circ h\right)\right)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} \left(g\left(f\circ h\right)\right)(b) \right] \Bigg\}.
\end{aligned}
$$

Multiplying the both sides $(\Gamma(\alpha))^{-1}$, we obtain (11) which complates the proof. $\square$

Remark 2.1. If we choose $h(x)=x$ in Lemma 2.1, then the inequality (11) reduces to (1.4).

Remark 2.2. If we choose $h(x)=x$, $g(x)=1$ and $\alpha=1$ in Lemma 2.1, we obtain Lemma 2.1 in [21].

Theorem 2.1. *Let $f: I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in X_h^p[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left| f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(h(a)) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(h(b))\right]\right.\\
&\left.\quad -\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\left(g\times(f\circ h)\right)(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\left(g\times(f\circ h)\right)(b)\right]\right|\\
&\le \frac{\|g\|_{X_h^{\infty}\left[a,\frac{a+b}{2}\right],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\left\{\left|f'(h(a))\right|\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+1}}{\alpha+1}(h(b)-h(a))\right.\right.\\
&\quad\left. -\left|f'(h(a))\right|\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\\
&\quad\left. +\left|f'(h(b))\right|\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\right\} + \frac{\|g\|_{X_h^{\infty}\left[\frac{a+b}{2},b\right],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\\
&\times\left\{\left|f'(h(b))\right|\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+1}}{\alpha+1}(h(b)-h(a)) - \frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right.\\
&\quad\left. +\left|f'(h(a))\right|\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right\}
\end{aligned} \tag{12}$$

with $\alpha>0$.

Proof. If $|f'|$ is convex on $[a,b]$, we know that for $t\in[a,b]$

$$\begin{aligned}
&\left|f'(h(t))\right|\\
&=\left|f'\left(\frac{h(b)-h(t)}{h(b)-h(a)}h(a)+\frac{h(t)-h(a)}{h(b)-h(a)}h(b)\right)\right|\\
&\le \frac{h(b)-h(t)}{h(b)-h(a)}\left|f'(h(a))\right| + \frac{h(t)-h(a)}{h(b)-h(a)}\left|f'(h(b))\right|.
\end{aligned}$$

From Lemma 2.1, we have

$$\left| f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(h(a)) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(h(b))\right]\right.$$

$$-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}\left(g\left(f\circ h\right)\right)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}\left(g\left(f\circ h\right)\right)(b)\right]\Bigg|$$

$$\leq\frac{1}{\Gamma(\alpha)}\left\{\int\limits_{a}^{\frac{a+b}{2}}\left|\int\limits_{a}^{t}(h(s)-h(a))^{\alpha-1}g(s)h'(s)\mathrm{d}s\right|\left|f'\left(h(t)\right)\right|h'(t)\mathrm{d}t\right.$$

$$\left.+\int\limits_{\frac{a+b}{2}}^{b}\left|\int\limits_{b}^{t}(h(b)-h(s))^{\alpha-1}g(s)h'(s)\mathrm{d}s\right|\left|f'\left(h(t)\right)\right|h'(t)\mathrm{d}t\right\}$$

$$\leq\frac{||g||_{X_h^{\infty}\left[a,\frac{a+b}{2}\right],\infty}}{(h(b)-h(a))\,\Gamma(\alpha)}\left\{\int\limits_{a}^{\frac{a+b}{2}}\left|\int\limits_{a}^{t}(h(s)-h(a))^{\alpha-1}h'(s)\mathrm{d}s\right|\right.$$

$$\times\left(h(b)-h(t)\left|f'\left(h(a)\right)\right|\right)h'(t)\mathrm{d}t$$

$$\left.+\int\limits_{a}^{\frac{a+b}{2}}\left|\int\limits_{a}^{t}(h(s)-h(a))^{\alpha-1}h'(s)ds\right|\left(h(t)-h(a)\left|f'\left(h(b)\right)\right|\right)h'(t)\mathrm{d}t\right\}$$

$$+\frac{||g||_{X_h^{\infty}\left[\frac{a+b}{2},b\right],\infty}}{(h(b)-h(a))\,\Gamma(\alpha)}\left\{\int\limits_{\frac{a+b}{2}}^{b}\left|\int\limits_{b}^{t}(h(b)-h(s))^{\alpha-1}h'(s)ds\right|\right.$$

$$\times\left(h(b)-h(t)\left|f'\left(h(a)\right)\right|\right)h'(t)\mathrm{d}t$$

$$\left.+\int\limits_{\frac{a+b}{2}}^{b}\left|\int\limits_{b}^{t}(h(b)-h(s))^{\alpha-1}h'(s)\mathrm{d}s\right|\left(h(t)-h(a)\left|f'\left(h(b)\right)\right|\right)h'(t)\mathrm{d}t\right\}.$$

$$\leq\frac{||g||_{X_h^{\infty}\left[a,\frac{a+b}{2}\right],\infty}}{(h(b)-h(a))\,\Gamma(\alpha+1)}\left\{\int\limits_{a}^{\frac{a+b}{2}}(h(t)-h(a))^{\alpha}\left(h(b)-h(t)\left|f'\left(h(a)\right)\right|\right)h'(t)\mathrm{d}t\right.$$

$$\left.+\int\limits_{a}^{\frac{a+b}{2}}(h(t)-h(a))^{\alpha+1}\left|f'\left(h(b)\right)\right|h'(t)\mathrm{d}t\right\}$$

$$+\frac{||g||_{X_h^{\infty}\left[\frac{a+b}{2},b\right],\infty}}{(h(b)-h(a))\,\Gamma(\alpha+1)}\left\{\int\limits_{\frac{a+b}{2}}^{b}(h(b)-h(t))^{\alpha+1}\left|f'\left(h(a)\right)\right|h'(t)\mathrm{d}t\right.$$

$$+\int_{\frac{a+b}{2}}^{b}(h(b)-h(t))^{\alpha-1}\left(h(t)-h(a)\left|f'(h(b))\right|\right)h'(t)\mathrm{d}t\Bigg\}$$

$$\leq\frac{||g||_{X_h^\infty[a,\frac{a+b}{2}],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\Bigg\{\left|f'(h(a))\right|\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+1}}{\alpha+1}(h(b)-h(a))\right.$$
$$\left.-\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]+\left|f'(h(b))\right|\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]\Bigg\}$$
$$+\frac{||g||_{X_h^\infty[\frac{a+b}{2},b],\infty}}{(h(b)-h(a))\Gamma(\alpha+1)}\Bigg\{\left|f'(h(b))\right|\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+1}}{\alpha+1}(h(b)-h(a))\right]$$
$$-\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}+\left|f'(h(a))\right|\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\Bigg\}.$$

This completes the proof. □

Remark 2.3. If we choose $h(x)=x$ in Theorem 2.1, we obtain Theorem 6 in [5].

Remark 2.4. If we choose $h(x)=x$, $g(x)=1$ and $\alpha=1$ in Theorem 2.1, we obtain Theorem 2.2 in [21].

Theorem 2.2. *Let $f:I\to\mathbb{R}$ be a differentiable mapping on I° and $f'\in X_h^p[a,b]$ with $a<b$ and $g:[a,b]\to\mathbb{R}$ is continuous. If $|f'|^q$ is convex on $[a,b]$, $q\geq 1$ then the following inequality for fractional integrals holds:*

$$f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}g(h(a))+J^\alpha_{\left(\frac{a+b}{2}\right)^+}g(h(b))\right]$$
$$-\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}(g\times(f\circ h))(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}(g\times(f\circ h))(b)\right]$$
$$\leq\frac{||g||_{X_h^\infty[a,\frac{a+b}{2}],\infty}}{[(h(b)-h(a))]^{\frac{1}{q}}\Gamma(\alpha)}\left(\frac{h\left(\frac{a+b}{2}\right)-h(a)}{\alpha(\alpha+1)}\right)^{1-\frac{1}{q}}\Bigg\{\left|f'(h(a))\right|^q \tag{13}$$
$$\times\left[\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+1}(h(b)-h(a))}{\alpha+1}-\frac{(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\right]$$
$$+\frac{|f'(h(b))|^q(h(\frac{a+b}{2})-h(a))^{\alpha+2}}{\alpha+2}\Bigg\}^{\frac{1}{q}}$$

$$+\frac{||g||_{X_h^\infty[\frac{a+b}{2},b],\infty}}{[(h(b)-h(a))]^{\frac{1}{q}}\Gamma(\alpha)}\left(\frac{h(b)-h(\frac{a+b}{2})}{\alpha(\alpha+1)}\right)^{1-\frac{1}{q}}\Bigg\{\left|f'(h(b))\right|^q$$

$$\times\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+1}(h(b)-h(a))}{\alpha+1}-\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]$$
$$\left.+\left|f'(h(a))\right|^{q}\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right\}^{\frac{1}{q}}.$$

with $\alpha>0$.

Proof. If $|f'|^q$ is convex on $[a,b]$, we know that for $t\in[a,b]$

$$\begin{aligned}\left|f'(h(t))\right|^{q}&=\left|f'\left(\frac{h(b)-h(t)}{h(b)-h(a)}h(a)+\frac{h(t)-h(a)}{h(b)-h(a)}h(b)\right)\right|^{q}\\&\le\frac{h(b)-h(t)}{h(b)-h(a)}\left|f'(h(a))\right|^{q}+\frac{h(t)-h(a)}{h(b)-h(a)}\left|f'(h(b))\right|^{q}.\end{aligned}$$

From Lemma 2.1, power mean inequality and the convexity of $|f'|^q$, it follows that

$$\begin{aligned}&f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}g(h(a))+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}g(h(b))\right]\\&\quad-\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}\left(g\left(f\circ h\right)\right)(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}\left(g\left(f\circ h\right)\right)(b)\right]\\&\le\frac{1}{\Gamma(\alpha)}\left\{\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(h(s)-h(a))^{\alpha-1}g(s)h'(s)\mathrm{d}s\right|h'(t)\mathrm{d}t\right)^{1-1/q}\right.\\&\qquad\left.\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(h(s)-h(a))^{\alpha-1}g(s)h'(s)\mathrm{d}s\right|\left|f'(h(t))\right|^{q}h'(t)\mathrm{d}t\right)^{1/q}\right\}\\&\quad+\frac{1}{\Gamma(\alpha)}\left\{\left(\int_{\frac{a+b}{2}}^{b}\left|\int_b^t(h(b)-h(s))^{\alpha-1}g(s)h'(s)\mathrm{d}s\right|h'(t)\mathrm{d}t\right)^{1-1/q}\right.\\&\qquad\left.\times\left(\int_{\frac{a+b}{2}}^{b}\left|\int_b^t(h(b)-h(s))^{\alpha-1}g(s)h'(s)\mathrm{d}s\right|\left|f'(h(t))\right|^{q}h'(t)\mathrm{d}t\right)^{1/q}\right\}\\&\le\frac{||g||_{X_h^\infty[a,\frac{a+b}{2}],\infty}}{\Gamma(\alpha)}\left\{\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t(h(s)-h(a))^{\alpha-1}h'(s)\mathrm{d}s\right|h'(t)\mathrm{d}t\right)^{1-1/q}\right.\end{aligned}$$

$$\times\left(\int_a^{\frac{a+b}{2}}\left|\int_a^t (h(s)-h(a))^{\alpha-1}h'(s)\mathrm{d}s\right|\left|f'(h(t))\right|^q h'(t)\mathrm{d}t\right)^{1/q}\Bigg\}$$

$$+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{\Gamma(\alpha)}\left\{\left(\int_{\frac{a+b}{2}}^{b}\left|\int_b^t (h(b)-h(s))^{\alpha-1}h'(s)\mathrm{d}s\right|h'(t)\mathrm{d}t\right)^{1-1/q}\right.$$

$$\left.\times\left(\int_{\frac{a+b}{2}}^{b}\left|\int_b^t (h(b)-h(s))^{\alpha-1}h'(s)\mathrm{d}s\right|\left|f'(h(t))\right|^q h'(t)\mathrm{d}t\right)^{1/q}\right\}.$$

$$\leq\frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{\left[(h(b)-h(a))\right]^{\frac{1}{q}}\Gamma(\alpha)}\left\{\int_a^{\frac{a+b}{2}}\left|\int_a^t (h(s)-h(a))^{\alpha-1}h'(s)\mathrm{d}s\right|\right.$$

$$\times\left(h(b)-h(t)\left|f'(h(a))\right|^q\right)h'(t)\mathrm{d}t$$

$$\left.+\int_a^{\frac{a+b}{2}}\left|\int_a^t (h(s)-h(a))^{\alpha-1}h'(s)ds\right|\left(h(t)-h(a)\left|f'(h(b))\right|^q\right)h'(t)\mathrm{d}t\right\}^{\frac{1}{q}}$$

$$+\frac{||g||_{X_h^\infty\left[\frac{a+b}{2},b\right],\infty}}{\left[(h(b)-h(a))\right]^{\frac{1}{q}}\Gamma(\alpha)}\left\{\int_{\frac{a+b}{2}}^{b}\left|\int_b^t (h(b)-h(s))^{\alpha-1}h'(s)\mathrm{d}s\right|\right.$$

$$\times\left(h(b)-h(t)\left|f'(h(a))\right|^q\right)h'(t)\mathrm{d}t$$

$$\left.+\int_{\frac{a+b}{2}}^{b}\left|\int_b^t (h(b)-h(s))^{\alpha-1}h'(s)\mathrm{d}s\right|\left(h(t)-h(a)\left|f'(h(b))\right|^q\right)h'(t)\mathrm{d}t\right\}^{\frac{1}{q}}$$

$$\leq\frac{||g||_{X_h^\infty\left[a,\frac{a+b}{2}\right],\infty}}{\left[(h(b)-h(a))\right]^{\frac{1}{q}}\Gamma(\alpha)}\left(\frac{h\left(\frac{a+b}{2}\right)-h(a)}{\alpha(\alpha+1)}\right)^{1-1/q}$$

$$\times\left\{\left[\frac{\left|f'(h(a))\right|^q\left(h(\frac{a+b}{2})-h(a)\right)^{\alpha+1}(h(b)-h(a))}{\alpha+1}\right.\right.$$

$$\left.\left.-\frac{\left|f'(h(a))\right|^q\left(h(\frac{a+b}{2})-h(a)\right)^{\alpha+2}}{\alpha+2}\right]+\left|f'(h(b))\right|^q\left[\frac{\left(h(\frac{a+b}{2})-h(a)\right)^{\alpha+2}}{\alpha+2}\right]\right\}^{\frac{1}{q}}$$

$$+\frac{||g||_{X_h^{\infty}\left[\frac{a+b}{2},b\right],\infty}}{\left[(h(b)-h(a))\right]^{\frac{1}{q}}\Gamma(\alpha)}\left(\frac{h(b)-h(\frac{a+b}{2})}{\alpha(\alpha+1)}\right)^{1-1/q}$$

$$\times\left\{\left|f'(h(b))\right|^q\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+1}(h(b)-h(a))}{\alpha+1}\right.\right.$$

$$\left.\left.-\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]+\left|f'(h(a))\right|^q\left[\frac{(h(b)-h(\frac{a+b}{2}))^{\alpha+2}}{\alpha+2}\right]\right\}^{\frac{1}{q}}. \qquad \square$$

Remark 2.5. If we choose $h(x)=x$ in Theorem 2.2, we obtain Theorem 7 in [5].

Lemma 2.2. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and let $g:[a,b]\to\mathbb{R}$. If f', $g\in X_h^p[a,b]$, then the following identity for fractional integrals holds:*

$$\begin{aligned}&\left\{f\left(h\left(\tfrac{a+b}{2}\right)\right)\left[-J_{a^+}^{\alpha}g(h(b))-J_{b^-}^{\alpha}g(h(a))\right.\right.\\&\qquad+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}g(h(b))+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}g(h(a))\\&\qquad\left.+\frac{1}{\Gamma(\alpha)}\int_a^b(h(b)-h(a))^{\alpha-1}g(s)h'(s)\mathrm{d}s\right]\\&+J_{a^+}^{\alpha}\left(g\left(f\circ h\right)\right)(b)+J_{b^-}^{\alpha}\left(g\left(f\circ h\right)\right)(a)\\&-J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}\left(g\left(f\circ h\right)\right)(b)-J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}\left(g\left(f\circ h\right)\right)(a)\\&\left.-\frac{1}{\Gamma(\alpha)}\int_a^b(h(b)-h(a))^{\alpha-1}g(t)f(h(t))h'(t)\mathrm{d}t\right\}\\&=\frac{1}{\Gamma(\alpha)}\int_a^b k(t)\mathrm{d}f(h(t))\end{aligned}\tag{14}$$

where

$$k(t)=\begin{cases}\displaystyle\int_a^t\Big[(h(b)-h(a))^{\alpha-1}-(h(b)-h(s))^{\alpha-1}\\ \qquad+(h(s)-h(a))^{\alpha-1}\Big]g(s)h'(s)\mathrm{d}s & t\in\left[a,\frac{a+b}{2}\right],\\ \displaystyle\int_b^t\Big[(h(b)-h(s))^{\alpha-1}-(h(s)-h(a))^{\alpha-1}\\ \qquad+(h(b)-h(a))^{\alpha-1}\Big]g(s)h'(s)\mathrm{d}s & t\in\left[\frac{a+b}{2},b\right].\end{cases}$$

Proof. It suffices to note that

$$\begin{aligned}
I &= \int_a^b k(t)\mathrm{d}f(h(t)) \\
&= \int_a^{\frac{a+b}{2}} \Bigg\{ \int_a^t \Big[(h(b)-h(a))^{\alpha-1} - (h(b)-h(s))^{\alpha-1} \\
&\qquad + (h(s)-h(a))^{\alpha-1} \Big] g(s)h'(s)\mathrm{d}s \Bigg\} \mathrm{d}f(h(t)) \\
&+ \int_{\frac{a+b}{2}}^b \Bigg\{ \int_b^t \Big[(h(b)-h(s))^{\alpha-1} - (h(s)-h(a))^{\alpha-1} \\
&\qquad + (h(b)-h(a))^{\alpha-1} \Big] g(s)h'(s)\mathrm{d}s \Bigg\} \mathrm{d}f(h(t)) \\
&= I_1 + I_2.
\end{aligned}$$

By integration by parts, we get

$$\begin{aligned}
I_1 &= \int_a^{\frac{a+b}{2}} \Bigg\{ \int_a^t \Big[(h(b)-h(a))^{\alpha-1} - (h(b)-h(s))^{\alpha-1} \\
&\qquad + (h(s)-h(a))^{\alpha-1} \Big] g(s)h'(s)\mathrm{d}s \Bigg\} \mathrm{d}f(h(t)) \\
&= \Bigg\{ \int_a^t \Big[(h(b)-h(a))^{\alpha-1} - (h(b)-h(s))^{\alpha-1} \\
&\qquad + (h(s)-h(a))^{\alpha-1} \Big] g(s)h'(s)\mathrm{d}s \Bigg\} \left. f(h(t)) \right|_a^{\frac{a+b}{2}} \\
&\qquad - \Bigg\{ \int_a^{\frac{a+b}{2}} \Big[(h(b)-h(a))^{\alpha-1} - (h(b)-h(t))^{\alpha-1} \\
&\quad + (h(t)-h(a))^{\alpha-1} \Big] g(t)f(h(t))h'(t)\mathrm{d}t \Bigg\} \\
&= f\left(h\left(\tfrac{a+b}{2}\right)\right) \Bigg\{ \int_a^{\frac{a+b}{2}} \Big[(h(b)-h(a))^{\alpha-1} - (h(b)-h(s))^{\alpha-1}. \\
&\qquad (h(s)-h(a))^{\alpha-1} \Big] g(s)h'(s)\mathrm{d}s \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&-\left\{\int_{a}^{\frac{a+b}{2}}\Big[(h(b)-h(a))^{\alpha-1}-(h(b)-h(t))^{\alpha-1}\right.\\
&\left.\qquad+(h(t)-h(a))^{\alpha-1}\Big]g(t)f(h(t))h'(t)\mathrm{d}t\right\}\\
&=\Gamma(\alpha)\Bigg[f\left(h\left(\tfrac{a+b}{2}\right)\right)\left\{J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}g(h(a))-J^{\alpha}_{a^{+}}g(h(b))\right.\\
&\left.\qquad+\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}(h(b)-h(a))^{\alpha-1}g(s)h'(s)\mathrm{d}s\right\}\\
&\qquad+J^{\alpha}_{a^{+}}\left(g\left(f\circ h\right)\right)(b)-J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}\left(g\left(f\circ h\right)\right)(a)\\
&\qquad-\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}(h(b)-h(a))^{\alpha-1}g(t)f(h(t))h'(t)\mathrm{d}t\Bigg]
\end{aligned}
$$

and similarly

$$
\begin{aligned}
I_2&=\int_{\frac{a+b}{2}}^{b}\left\{\int_{b}^{t}\Big[(h(b)-h(s))^{\alpha-1}-(h(s)-h(a))^{\alpha-1}\right.\\
&\left.\qquad+(h(b)-h(a))^{\alpha-1}\Big]g(s)h'(s)\mathrm{d}s\right\}\mathrm{d}f(h(t))\\
&=\left\{\int_{b}^{t}\Big[(h(b)-h(s))^{\alpha-1}-(h(s)-h(a))^{\alpha-1}\right.\\
&\left.\qquad+(h(b)-h(a))^{\alpha-1}\Big]g(s)h'(s)\mathrm{d}s\right\}f(h(t))\Big|_{\frac{a+b}{2}}^{b}\\
&-\left\{\int_{\frac{a+b}{2}}^{b}\Big[(h(b)-h(t))^{\alpha-1}-(h(t)-h(a))^{\alpha-1}\right.\\
&\left.\qquad+(h(b)-h(a))^{\alpha-1}\Big]g(t)f(h(t))h'(t)\mathrm{d}t\right\}\\
&=f\left(h\left(\tfrac{a+b}{2}\right)\right)\left\{\int_{\frac{a+b}{2}}^{b}\Big[(h(b)-h(s))^{\alpha-1}-(h(s)-h(a))^{\alpha-1}\right.\\
&\left.\qquad+(h(b)-h(a))^{\alpha-1}\Big]g(s)h'(s)\mathrm{d}s\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\left\{\int_{\frac{a+b}{2}}^{b}\Big[(h(b)-h(t))^{\alpha-1}-(h(t)-h(a))^{\alpha-1}\right.\\
&\qquad\left.+(h(b)-h(a))^{\alpha-1}\Big]g(t)f(h(t))h'(t)\mathrm{d}t\right\}\\
&=\Gamma(\alpha)\Bigg[f\left(h\left(\tfrac{a+b}{2}\right)\right)\Bigg\{J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}g(h(b))-J^{\alpha}_{b^{-}}g(h(a))\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}(h(b)-h(a))^{\alpha-1}g(s)h'(s)\mathrm{d}s\Bigg\}\\
&\qquad-\Bigg\{J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}\left(g\left(f\circ h\right)\right)(b)+J^{\alpha}_{b^{-}}\left(g\left(f\circ h\right)\right)(a)\\
&\qquad-\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}(h(b)-h(a))^{\alpha-1}g(t)f(h(t))h'(t)\mathrm{d}t\Bigg\}\Bigg].
\end{aligned}
$$

Thus, can write

$$
\begin{aligned}
I_1+I_2=\Gamma(\alpha)\Bigg\{&f\left(h\left(\tfrac{a+b}{2}\right)\right)\Bigg[-J^{\alpha}_{a^{+}}g(h(b))-J^{\alpha}_{b^{-}}g(h(a))\\
&+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}g(h(b))+J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}g(h(a))\\
&+\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}(h(b)-h(a))^{\alpha-1}g(s)h'(s)\mathrm{d}s\\
&+\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}(h(b)-h(a))^{\alpha-1}g(s)h'(s)\mathrm{d}s\Bigg]\\
&+J^{\alpha}_{b^{-}}\left(g\left(f\circ h\right)\right)(a)+J^{\alpha}_{a^{+}}\left(g\left(f\circ h\right)\right)(b)\\
&-J^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}\left(g\left(f\circ h\right)\right)(a)-J^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}\left(g\left(f\circ h\right)\right)(b)\\
&-\frac{1}{\Gamma(\alpha)}\int_{a}^{\frac{a+b}{2}}(h(b)-h(a))^{\alpha-1}g(t)f(h(t))h'(t)\mathrm{d}t\\
&-\frac{1}{\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}(h(b)-h(a))^{\alpha-1}g(t)f(h(t))h'(t)\mathrm{d}t\Bigg\}
\end{aligned}
$$

Multiplying the both sides $(\Gamma(\alpha))^{-1}$, we obtain (14) which completes the proof. □

Remark 2.6. If we choose $h(x) = x$ in Lemma 2.2, we have

$$\Big\{ f\left(\tfrac{a+b}{2}\right) \Big[-J_{a^+}^{\alpha} g(b) - J_{b^-}^{\alpha} g(a) + J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} g(b) + J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} g(a) + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b g(s) \mathrm{d}s \Big] + J_{a^+}^{\alpha} (gf)(b) + J_{b^-}^{\alpha} (gf)(a) - J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} (gf)(b) - J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} (gf)(a) - \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b g(t) f(t) \mathrm{d}t \Big\}$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^b k(t) \mathrm{d}f(t)$$

where

$$k(t) = \begin{cases} \int_a^t \left[(b-a))^{\alpha-1} - (b-s)^{\alpha-1} + (s-a)^{\alpha-1} \right] g(s) \mathrm{d}s, & t \in \left[a, \frac{a+b}{2}\right], \\ \int_b^t \left[(b-s)^{\alpha-1} - (s-a)^{\alpha-1} + (b-a)^{\alpha-1} \right] g(s) \mathrm{d}s, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Remark 2.7. If we choose $h(x) = x$ and $g(x) = 1$ in Lemma 2.2, we have

$$\Big\{ f\left(\tfrac{a+b}{2}\right) \left[(b-a)^{\alpha} \left(1 + \frac{1}{\alpha} \left(\frac{1}{2^{\alpha}} - 2 \right) \right) \right] + J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) - J_{\left(\frac{a+b}{2}\right)^+}^{\alpha} f(b) - J_{\left(\frac{a+b}{2}\right)^-}^{\alpha} f(a) - \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^b f(t) \mathrm{d}t \Big\}$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^b k(t) \mathrm{d}f(t)$$

where

$$k(t) = \begin{cases} \int_a^t \left[(b-a)^{\alpha-1} - (b-s)^{\alpha-1} + (s-a)^{\alpha-1} \right] \mathrm{d}s, & t \in \left[a, \frac{a+b}{2}\right], \\ \int_b^t \left[(b-s)^{\alpha-1} - (s-a)^{\alpha-1} + (b-a)^{\alpha-1} \right] \mathrm{d}s, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Remark 2.8. If we choose $h(x) = x$, $\alpha = 1$ and $g(x) = 1$ in Lemma 2.2, we obtain Lemma 2.1 in [21]

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t - f\left(\tfrac{a+b}{2}\right)$$
$$= (b-a)\left(\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)\mathrm{d}t + \int_{\frac{1}{2}}^1 (t-1)\, f'(ta+(1-t)b)\mathrm{d}t\right).$$

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in X_h^p[a,b]$ with $a < b$ and $g : [a,b] \to \mathbb{R}$ is continuous. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\Bigg\{ f\left(\tfrac{a+b}{2}\right)\Big[-J^\alpha_{a^+}g(b) - J^\alpha_{b^-}g(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+}g(b)$$
$$+ J^\alpha_{\left(\frac{a+b}{2}\right)^-}g(a) + \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b g(s)\mathrm{d}s\Big]$$
$$+ J^\alpha_{a^+}(gf)(b) + J^\alpha_{b^-}(gf)(a) - J^\alpha_{\left(\frac{a+b}{2}\right)^+}(gf)(b)$$
$$- J^\alpha_{\left(\frac{a+b}{2}\right)^-}(gf)(a) - \frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b g(t)f(t)\mathrm{d}t\Bigg\}$$
$$\leq \frac{||g||_{\left[a,\frac{a+b}{2}\right],\infty}}{(b-a)\Gamma(\alpha)}\Bigg\{\int_a^{\frac{a+b}{2}}\Big[\int_a^t\left[(b-a)^{\alpha-1} - (b-s)^{\alpha-1} + (s-a)^{\alpha-1}\right]\mathrm{d}s\Big]$$
$$\times\left((b-t)\left|f'(a)\right| + (t-a)\left|f'(b)\right|\right)\mathrm{d}t\Bigg\}$$
$$+ \frac{||g||_{\left[\frac{a+b}{2},b\right],\infty}}{(b-a)\Gamma(\alpha)}\Bigg\{\int_{\frac{a+b}{2}}^b\Bigg[\int_b^t\left[(b-s)^{\alpha-1} - (s-a)^{\alpha-1} + (b-a)^{\alpha-1}\right]g(s)\mathrm{d}s\Bigg]$$
$$\times\left((b-t)\left|f'(a)\right| + (t-a)\left|f'(b)\right|\right)\mathrm{d}t\Bigg\}$$

with $\alpha > 0$.

Proof. If $|f'|$ is convex on $[a,b]$, we know that for $t \in [a,b]$

$$\left|f'(t)\right| = \left|f'\left(\frac{b-t}{b-a}a + \frac{t-a}{b-a}b\right)\right| \leq \frac{b-t}{b-a}\left|f'(a)\right| + \frac{t-a}{b-a}\left|f'(b)\right|.$$

From Remark 2.6, we have

$$\Big\{f\left(\tfrac{a+b}{2}\right)\Big[-J_{a^+}^{\alpha}g(b)-J_{b^-}^{\alpha}g(a)+J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}g(b)+J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}g(a)$$

$$+\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b g(s)\mathrm{d}s\Big]+J_{a^+}^{\alpha}(gf)(b)+J_{b^-}^{\alpha}(gf)(a)$$

$$-J_{\left(\frac{a+b}{2}\right)^+}^{\alpha}(gf)(b)-J_{\left(\frac{a+b}{2}\right)^-}^{\alpha}(gf)(a)-\frac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}\int_a^b g(t)f(t)\mathrm{d}t\Big\}$$

$$\leq\frac{1}{\Gamma(\alpha)}\Big\{\int_a^{\frac{a+b}{2}}\left|\int_a^t\left[(b-a)^{\alpha-1}-(b-s)^{\alpha-1}+(s-a)^{\alpha-1}\right]g(s)\mathrm{d}s\right|\left|f'(t)\right|\mathrm{d}t$$

$$+\int_{\frac{a+b}{2}}^b\left|\int_b^t\left[(b-s)^{\alpha-1}-(s-a)^{\alpha-1}+(b-a)^{\alpha-1}\right]\mathrm{d}s\right|g(s)\left|f'(t)\right|\mathrm{d}t\Big\}$$

$$\leq\frac{1}{(b-a)\Gamma(\alpha)}\Big\{\int_a^{\frac{a+b}{2}}\left[\int_a^t\left[(b-a)^{\alpha-1}-(b-s)^{\alpha-1}+(s-a)^{\alpha-1}\right]g(s)\mathrm{d}s\right]$$

$$\times\left((b-t)\left|f'(a)\right|+(t-a)\left|f'(b)\right|\right)\mathrm{d}t$$

$$+\int_{\frac{a+b}{2}}^b\left[\int_b^t\left[(b-s)^{\alpha-1}-(s-a)^{\alpha-1}+(b-a)^{\alpha-1}\right]g(s)\mathrm{d}s\right]$$

$$\times\left((b-t)\left|f'(a)\right|+(t-a)\left|f'(b)\right|\right)\mathrm{d}t\Big\}$$

$$\leq\frac{||g||_{\left[a,\frac{a+b}{2}\right],\infty}}{(b-a)\Gamma(\alpha)}\Big\{\int_a^{\frac{a+b}{2}}\left[\int_a^t\left[(b-a)^{\alpha-1}-(b-s)^{\alpha-1}+(s-a)^{\alpha-1}\right]\mathrm{d}s\right]$$

$$\times\left((b-t)\left|f'(a)\right|+(t-a)\left|f'(b)\right|\right)\mathrm{d}t\Big\}$$

$$+\frac{||g||_{\left[\frac{a+b}{2},b\right],\infty}}{(b-a)\Gamma(\alpha)}\Big\{\int_{\frac{a+b}{2}}^b\left[\int_b^t\left[(b-s)^{\alpha-1}-(s-a)^{\alpha-1}+(b-a)^{\alpha-1}\right]g(s)\mathrm{d}s\right]$$

$$\times\left((b-t)\left|f'(a)\right|+(t-a)\left|f'(b)\right|\right)\mathrm{d}t\Big\}$$

$$\leq \frac{||g||_{\left[a,\frac{a+b}{2}\right],\infty}}{(b-a)\Gamma(\alpha)}\left\{\int_a^{\frac{a+b}{2}}\left[\int_a^t\left[(b-a)^{\alpha-1}-(b-s)^{\alpha-1}+(s-a)^{\alpha-1}\right]\mathrm{d}s\right]\right.$$
$$\left.\times\left((b-t)\left|f'(a)\right|+(t-a)\left|f'(b)\right|\right)\mathrm{d}t\right\}$$
$$+\frac{||g||_{\left[\frac{a+b}{2},b\right],\infty}}{(b-a)\Gamma(\alpha)}\left\{\int_{\frac{a+b}{2}}^{b}\left[\int_b^t\left[(b-s)^{\alpha-1}-(s-a)^{\alpha-1}+(b-a)^{\alpha-1}\right]g(s)\mathrm{d}s\right]\right.$$
$$\left.\times\left((b-t)\left|f'(a)\right|+(t-a)\left|f'(b)\right|\right)\mathrm{d}t\right\}.$$

This completes the proof. □

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